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# Exceptional geometry and tensor fields

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Abstract: We present a tensor calculus for exceptional generalised geometry. Expressions for connections, torsion and curvature are given a unified formulation for different exceptional groups  $E_{n(n)}$ . We then consider “tensor gauge fields” coupled to the exceptional generalised gravity. Many of the properties of forms on manifolds are carried over to these fields.

## 1. INTRODUCTION

The dualities of string theory or M-theory treat momenta and brane charges on an equal footing. By generalising space-time to include directions conjugate to brane charges, such symmetries can be made manifest, but obviously the concept of geometry has to be modified. There has been considerable progress in the understanding of such models recently, both in the context of U-duality [1-4], which is the main focus of the present paper, and T-duality. We refer to both types of theories as “generalised geometry”; doubled geometry [5-22] in the case of T-duality, and exceptional geometry [23-35] in the case of U-duality.

Turning to the state of the subject of exceptional geometry, it has been shown that it is possible to formulate the dynamics of a generalised metric, parametrising a coset  $G/H$  with  $G = E_{n(n)} \times \mathbb{R}^+$  and  $H$  its maximal compact subgroup, in a manner which respects local symmetries, generalising and including diffeomorphisms [25-29,32,34]. There are also results on an underlying geometry and tensor formalism [32,35], but the covariant tensor calculus has so far been limited to  $n = 4$  [35].

The purpose of the present paper is twofold. We give a universal (*i.e.*, valid for all  $n \leq 7$ ) version of exceptional geometry, and a tensor formalism that agrees with the one given for  $n = 4$  [35] and makes manifest the symmetry of ref. [32]. We also initiate an investigation of what may be thought of as differential geometry on a generalised manifold. A sequence of  $G$  modules, in many respect analogous to forms on ordinary manifolds, are given, and we describe how they may accommodate tensor (non-gravitational) gauge fields.

The paper is organised as follows. After some background on exceptional geometry in Section 2, we turn to the covariant construction of the generalised geometry in terms of vielbeins, connections and curvature in Sections 3-5. Section 6 deals with the dynamics of tensor fields coupled to generalised geometry. We summarise and point out some interesting questions in the concluding Section. Some conventions are given in an Appendix.

## 2. PRELIMINARIES ON EXCEPTIONAL GEOMETRY

As mentioned in the Introduction, we are concerned with a generalisation of geometry, where the traditional rôle of  $GL(n)$  in ordinary geometry is subsumed by the group  $G = E_{n(n)} \times \mathbb{R}^+$ , and that of the locally realised rotation group by the maximal compact subgroup  $H \subset G$ .

Generalised momenta transform in a module  $\overline{R}_1$  of  $G$ . A central identity in generalised geometry is the section condition. It states that bilinears in momenta projected on a certain module of  $G$ ,  $\overline{R}_2$ , vanish. Although this condition is  $G$ -covariant, its solutions effectively single out  $n$  directions on which fields may depend.

It is well known how to form a generalised Lie derivative, governing the generalised diffeomorphisms, which effectively include tensor gauge transformation in addition to ordinary diffeomorphisms. The generalised diffeomorphisms, acting on a vector, take the form

$$\mathcal{L}_U V^M = L_U V^M + Y^{MN}{}_{PQ} \partial_N U^P V^Q \quad (2.1)$$

( $L_U$  being the ordinary Lie derivative), which can be rewritten as

$$\begin{aligned} \mathcal{L}_U V^M &= U^N \partial_N V^M - \alpha P_{(\text{adj})}{}^M{}_{N, P}{}^Q \partial_P U^Q V^N + \beta \partial_N U^N V^M \\ &= U^N \partial_N V^M + Z^{MN}{}_{PQ} \partial_N U^P V^Q , \end{aligned} \quad (2.2)$$

where  $P_{(\text{adj})}$  projects on the adjoint of  $E_{n(n)}$  (we constrain the analysis to  $n \geq 4$ , where this group is simple). For  $n \leq 6$ , the tensor  $Y$  is proportional to the projection on  $R_2$ ,

$$Y^{MN}{}_{PQ} = 2(n-1) P_{(R_2)}^{MN}{}_{PQ} , \quad (2.3)$$

and for  $n = 7$  it contains an additional antisymmetric term  $\frac{1}{2} \varepsilon^{MN} \varepsilon_{PQ}$ . The constants  $\alpha_n$  take the values 3, 4, 6, 12 for  $n = 4, 5, 6, 7$ , respectively, while  $\beta_n = \frac{1}{9-n}$ .

The closure of the algebra of generalised diffeomorphisms relies on certain identities involving the invariant tensor  $Y$ . The simplest of these is the section condition itself,

$$Y^{MN}{}_{PQ} \partial_M \otimes \partial_N = 0 , \quad (2.4)$$

where the  $\otimes$  sign signifies that the two derivatives may act on any pair of fields. Another important identity is the nonlinear relation

$$(Y^{MN}{}_{TQ} Y^{TP}{}_{RS} - Y^{MN}{}_{RS} \delta_Q^P) \partial_{(N} \otimes \partial_{P)} = 0 , \quad (2.5)$$

which can also be written

$$(Z^{MN}{}_{TQ} Z^{TP}{}_{RS} + Z^{MP}{}_{RQ} \delta_S^N) \partial_{(N} \otimes \partial_{P)} = 0 . \quad (2.6)$$

Notice, that while eq. (2.5) manifests the  $R_2$  and  $\overline{R}_2$  projections of the index pairs  ${}^{MN}$  and  ${}_{RS}$ , the form (2.6) manifests the  $\mathfrak{g}$  projections in the pairs  ${}^M{}_Q$  and  ${}^P{}_R$ .

The parameters of generalised diffeomorphisms come in  $R_1$ , and it was demonstrated in ref. [34] that the infinite sequences  $\{R_k\}$  are responsible for the reducibility of the transformations. As we will see in Section 6, part of the sequence has many properties in common with forms in ordinary geometry, which is how we will be able to use them for constructing tensor fields. Before that is possible, we need to develop a tensor formalism.

$n$	$R_1$	$R_2$	$R_3$	$R_4$	$R_5$
3	$(\mathbf{3}, \mathbf{2})$	$(\overline{\mathbf{3}}, \mathbf{1})$	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$	$(\overline{\mathbf{3}}, \overline{\mathbf{2}})$
4	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\mathbf{5}$	$\overline{\mathbf{10}}$	$\mathbf{24}$
5	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	
6	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$		
7	$\mathbf{56}$	$\mathbf{133}$			

Table 1: A partial list of modules  $R_k^{(n)}$ .

### 3. TENSORS AND CONNECTIONS

The property (2.2) of the generalised Lie derivative on vectors ensures that it can be defined on a tensor carrying an arbitrary number of indices in  $R_1$  and  $\overline{R}_1$ , with the transformation

$$\begin{aligned}
 \mathcal{L}_U W^{M_1 \dots M_p}_{N_1 \dots N_q} &= U^P \partial_P W^{M_1 \dots M_p}_{N_1 \dots N_q} \\
 &+ \sum_{i=1}^p Z^{M_i Q}_{RP} \partial_Q U^R W^{M_1 \dots M_{i-1} P M_{i+1} \dots M_p}_{N_1 \dots N_q} \\
 &- \sum_{i=1}^q Z^{PQ}_{RN_i} \partial_Q U^R W^{M_1 \dots M_p}_{N_1 \dots N_{i-1} P N_{i+1} \dots N_q} ,
 \end{aligned} \tag{3.1}$$

so that tensor products and contractions respect the tensorial property.

Note that composition of tensors implies that the  $\mathbb{R}$ -weight is not freely assigned. Not any invariant  $E_{n(n)}$  tensor is a tensor under generalised diffeomorphisms. For example,  $E_6$  has an invariant tensor  $c^{MNP}$ . In order to be a tensor under generalised diffeomorphisms it would need to carry total  $\mathbb{R}$  weight 3, if the weight of a vector is normalised to one. Otherwise it becomes a tensor density. On the other hand,  $c^{MNP} c_{QRS}$  is a tensor.

We will introduce an affine connection,  $\Gamma_{MN}^P$ . As matrices  $(\Gamma_M)_N^P$ ,  $\Gamma_M$  are valued in the Lie algebra  $\mathfrak{g} = \mathfrak{e}_{n(n)} \oplus \mathbb{R}$ . Note that this excludes any specific symmetry properties for

the lower indices. Defining a covariant derivative  $D = \partial + \Gamma$ , the transformation rule of the connection should ensure that  $D_M W^{\{N\}}_{\{P\}}$  is a tensor if  $W^{\{N\}}_{\{P\}}$  is a tensor. We use the convention

$$\begin{aligned} D_M V_N &= \partial_M V_N + \Gamma_{MN}{}^P V_P , \\ D_M V^N &= \partial_M V^N - \Gamma_{MP}{}^N V^P , \end{aligned} \tag{3.2}$$

with the obvious generalisation to arbitrary number of indices.

The covariant derivatives of eq. (3.2) are valid for tensors, *i.e.*, for objects where each  $R_1$  index is accompanied with a certain  $\mathbb{R}$ -weight  $w$ , which we may normalise to 1, and accordingly  $-1$  for each  $\overline{R}_1$  index. This is not always an ideal way of describing modules. One may for example want to use invariant tensors of  $E_{n(n)}$  which do not have weight zero. One example is the duality  $R_k \leftrightarrow \overline{R}_{9-n-k}$ . It may sometimes be more convenient to represent, say,  $R_{8-n} = \overline{R}_1$  with one lower index instead of  $8-n$  upper ones. This amounts to considering “tensor densities”, by specifying  $E_{n(n)}$  module and  $\mathbb{R}$ -weight  $w$ . There is no acute need of distinguishing tensors and “tensor densities”, and we will use the term “tensor” for both. The covariant derivatives (taking a tensor of weight  $w$  to one of weight  $w-1$ ) on vectors and covectors, with natural generalisation to arbitrary index structures, are

$$\begin{aligned} D_M W_N &= \partial_M W_N + \Gamma_{MN}{}^P W_P - \frac{w+1}{|R_1|} \Gamma_{MP}{}^P W_N , \\ D_M V^N &= \partial_M V^N - \Gamma_{MP}{}^N V^P - \frac{w-1}{|R_1|} \Gamma_{MP}{}^P V^N . \end{aligned} \tag{3.3}$$

Demanding that the covariant derivative takes tensors to tensors immediately leads to the transformation rule for the connection,

$$\begin{aligned} \delta_\xi \Gamma_{MN}{}^P &= \mathcal{L}_\xi \Gamma_{MN}{}^P + Z^{PQ}{}_{RN} \partial_M \partial_Q \xi^R \\ &= \mathcal{L}_\xi \Gamma_{MN}{}^P - \partial_M \partial_N \xi^P + Y^{PQ}{}_{RN} \partial_M \partial_Q \xi^R . \end{aligned} \tag{3.4}$$

As mentioned, the generic  $E_{n(n)}$  module for the affine connection is  $\overline{R}_1 \otimes \mathfrak{g}$ . Not all of the irreducible components of  $\Gamma$  can appear in the inhomogeneous terms of eq. (3.4). Only the part occurring in  $(\vee^2 \overline{R}_1 \ominus \overline{R}_2) \otimes R_1$  will pick up inhomogeneous transformation terms. We define:

*Torsion is defined as the irreducible modules in the affine connection transforming homogeneously, i.e., with the generalised Lie derivative.*

Defined in this covariant way, torsion can consistently be set to zero.

It is quite straightforward to verify that the overlap  $[\overline{R}_1 \otimes \mathfrak{g}] \cap [(\vee^2 \overline{R}_1 \ominus \overline{R}_2) \otimes R_1]$  generically consists of a small module, which is  $\overline{R}_1$ , and a big module, which is the largest module in the product of  $\overline{R}_1$  and the adjoint. The torsion module, which is the rest of  $\Gamma$ ,

consists of a small module  $\overline{R}_1$  and a bigger one (reducible for low  $n$ ), which turns out to coincide with  $\overline{R}_{10-n}$ <sup>1</sup>.

$n$	torsion	non-torsion
3	$2(\overline{3}, 2) \oplus (6, 2)$	$(\overline{3}, 2) \oplus (\overline{3}, 4) \oplus (15, 2)$
4	$\overline{10} \oplus \overline{15} \oplus 40$	$\overline{10} \oplus \overline{175}$
5	$\overline{16} \oplus 144$	$\overline{16} \oplus \overline{560}$
6	$\overline{27} \oplus 351'$	$\overline{27} \oplus \overline{1728}$
7	$56 \oplus 912$	$56 \oplus 6480$

Table 2: Torsion and non-torsion part of the affine connection.

We need explicit expressions for the torsion, or expressions that a torsion-free connection satisfies. It turns out that

$$T_{MN}{}^P = \Gamma_{MN}{}^P + Z^{PQ}{}_{RN} \Gamma_{QM}{}^R \quad (3.5)$$

transforms as a tensor. This is verified by direct insertion into the transformation rule (3.4) and use of the identity (2.6). A torsion-free connection obeys

$$\Gamma_{MN}{}^P + Z^{PQ}{}_{RN} \Gamma_{QM}{}^R = 0 , \quad (3.6)$$

or, equivalently,  $2\Gamma_{[MN]}{}^P + Y^{PQ}{}_{RN} \Gamma_{QM}{}^R = 0$ . Note that the result from ordinary geometry is recovered for  $Y = 0$ .

It is straightforward to take a trace to determine which combination of the two  $\overline{R}_1$ 's is torsion and which is torsion-free. Contracting eq. (3.6) with  $\delta_P^N$  and using  $Z^{MP}{}_{PN} = \frac{|R_1|}{9-n} \delta_N^M$  shows that a torsion-free connection satisfies

$$\Gamma_{MN}{}^N + \frac{|R_1|}{9-n} \Gamma_{NM}{}^N = 0 . \quad (3.7)$$

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<sup>1</sup> It has been observed in ref. [32] that this torsion module can be identified with the embedding tensor of gauged supergravity. Work by Palmkvist [36] identifies a new class of algebras, symmetric under  $R_p \rightarrow \overline{R}_{9-n-p}$  where torsion appears as  $R_{-1}$ .

On the other hand, contracting eq. (3.6) with  $\delta_P^M$  gives

$$Y_{MN}{}^{QR}\Gamma_{QR}{}^N = -2\Gamma_{[NM]}{}^N = -(1 + \frac{|R_1|}{9-n})\Gamma_{NM}{}^N . \quad (3.8)$$

For  $n < 7$  this identity may be used to derive a “stronger” constraint. Since  $Y_{MN}{}^{QR}\Gamma_{QR}{}^P$  can only contain the  $\overline{R}_1$  part of a torsion-free connection<sup>2</sup>, it must be proportional to  $Y_{MN}{}^{PQ}\Gamma_{RQ}{}^R$ , and the proportionality constant is determined from eq. (3.8). The resulting relation is

$$Y_{MN}{}^{QR}\Gamma_{QR}{}^P + Y_{MN}{}^{PQ}\Gamma_{RQ}{}^R = 0 . \quad (3.9)$$

This relation is useful for determining when covariant derivatives are connection-free; see below.

The generalised Lie derivative on a vector does not contain any non-homogeneously transforming connection, if one replaces the naked derivatives with covariant ones. This is verified by replacing the derivatives in  $\mathcal{L}_U V$  of eq. (2.2) with covariant derivatives and checking that the connections come in the torsion combination of eq. (3.6). This property was used as a definition of torsion (equivalent to ours) in ref. [32].

Eq. (3.6) contains the torsion modules in the connection. The actual torsion-free connection cannot be obtained simply by adding a multiple of  $T_{MN}{}^P$  to  $\Gamma_{MN}{}^P$ , since the different torsion modules take different eigenvalues under  $\Gamma \rightarrow T$ .

#### 4. VIELBEINS AND COMPATIBLE CONNECTIONS

The structure group  $G = E_{n(n)} \times \mathbb{R}^+$  has a locally realised subgroup  $H$ , which in the signature we are using is the maximal compact subgroup  $H = K(E_{n(n)})$ . We denote  $R_1$  indices under  $H$  by  $A, B, \dots$

Consider a vielbein (frame field)  $E_M{}^A$ , which is a group element of  $E_{n(n)} \times \mathbb{R}^+$ . Locally it represents an element of the coset  $G/H$ , so it should be considered modulo local  $H$ -transformations from the right. It can be used to form a metric  $G_{MN} = E_M{}^A E_N{}^B \delta_{AB}$ , where  $\delta_{AB}$  is an  $H$ -invariant constant metric.

We want to impose that the vielbein is covariantly constant, when transported by a covariant derivative containing both affine and spin connections:

$$D_M E_N{}^A = \partial_M E_N{}^A + \Gamma_{MN}{}^P E_P{}^A - E_N{}^B \Omega_{MB}{}^A = 0 . \quad (4.1)$$

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<sup>2</sup> Because  $\overline{R}_2 \otimes R_1$  does not contain the big torsion-free connection module. This is not true for  $n = 7$ , where  $\overline{R}_2$  is the adjoint.

We now want to examine to what extent the connections are determined from the vanishing of torsion together with the compatibility equation (4.1). The affine connection can be eliminated from the equation by the use of the vanishing torsion condition — this simply amounts to forming a combination of eq. (4.1) that contains  $\Gamma$  through  $T$  of eq. (3.5). The result is

$$(D_M^{(\Omega)} EE^{-1})_N{}^P + Z^{PQ}{}_{RN} (D_Q^{(\Omega)} EE^{-1})_M{}^R = 0 . \quad (4.2)$$

On the other hand, the spin connection can be eliminated by projecting the compatibility equation on its  $\mathfrak{g}/\mathfrak{h}$  part. Note that when we talk about the local subgroup  $H$  we always mean the one defined by the vielbein. The projection is easy, since after lowering one index, the symmetric part of  $\mathfrak{g}$  is  $\mathfrak{g}/\mathfrak{h}$  and the antisymmetric part  $\mathfrak{h}$ . This leads to

$$(E^{-1} D_M^{(\Gamma)} E)_{(AB)} = 0 , \quad (4.3)$$

or, equivalently,

$$D_M G_{NP} = \partial_M G_{NP} + 2\Gamma_{M(NP)} = 0 . \quad (4.4)$$

To analyse the compatibility equations for the spin connection (4.2) and the affine connection (4.3), one must decompose into  $H$ -modules. One then finds that the content of eq. (4.2), which is identical to the torsion modules of Table 2, is smaller than the content of  $\Omega$ , which is  $R_1 \otimes \mathfrak{h}$ . The missing module  $\Sigma$  is the “big” irreducible module in  $R_1 \otimes \mathfrak{h}$ , *i.e.*, the  $H$ -module whose highest weight is the sum of the highest weights of  $R_1$  and  $\mathfrak{h}$ . Similarly, the same result is obtained from the compatibility for the affine connection, so there is always an undefined part (in the same module) of a torsion-free compatible affine connection. This is summarised in the table below, whose content agrees with ref. [32].

$n$	$H$	undetermined connection $\Sigma$
4	$SO(5)$	<b>35</b> = (04)
5	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$	<b>(4, 20)</b> $\oplus$ <b>(20, 4)</b> = (01)(03) $\oplus$ (03)(01)
6	$USp(8)/\mathbb{Z}_2$	<b>594</b> = (2100)
7	$SU(8)/\mathbb{Z}_2$	<b>1280</b> $\oplus$ <b><math>\overline{1280}</math></b> = (1100001) $\oplus$ (1000011)

Table 3: The undetermined part of a compatible torsion-free connection

This means, that if connection is not to represent independent degrees of freedom, one should only introduce covariant derivatives mapping between certain special pairs of modules. Consider two modules  $U$  and  $V$  under  $H$  (or its double cover), and let a covariant



derivative map from one to the other. This means that  $R_1 \otimes U \supset V$ . We are then only allowed to do this for pairs where at the same time  $\Sigma \otimes U \not\supset V$ . Some such pairs (“spinor” and “gravitino” modules) were discussed in refs. [32,33], and we will encounter other ones later.

A special case of such well-defined covariant derivatives consists of situations where not only the  $\Sigma$  part of a connection is absent, but where connection is altogether absent, and a covariant derivative equals an ordinary derivative. Such connection-free actions of derivatives will be important for our description of tensor gauge fields in Section 6, but we will already at this point check what the weight of a vector  $W^M$  must be in order for the divergence  $D_M W^M$  to be connection-free. From eq. (3.3) it follows that

$$D_M W^M = \partial_M W^M - \Gamma_{MN}{}^M W^N + \frac{w-1}{9-n} \Gamma_{NM}{}^N W^M . \quad (4.5)$$

The connection terms cancel for  $w = 10 - n$ , which can be expressed as

$$|G|^{-\frac{9-n}{2|R_1|}} D_M V^M = \partial_M (|G|^{-\frac{9-n}{2|R_1|}} V^M) \quad (4.6)$$

for a vector of weight 1. This result will have bearing on any discussion on measures and partial integration.

At this point, we would also like to comment on the relation between the present approach and the one used in a recent paper by Park and Suh [35]. There, the affine connection is subject to precisely the right number of constraints to make it uniquely determined from compatibility. In addition to the torsion condition, this procedure amounts to setting, by hand, the  $\Sigma$  module in  $\Gamma$  to zero. The resulting derivative with connection is then not fully covariant, but will behave as such acting between certain modules, the pairs described in the previous paragraph. We tend to prefer the present, geometric description, which allows for connections to transform as such (both with respect to generalised diffeomorphisms and local  $H$  transformations).

## 5. CURVATURE

We will now examine how curvature can be defined. We write the transformation rule (3.4) for the affine connection as

$$\Delta_\xi \Gamma_{MN}{}^P \equiv (\delta_\xi - \mathcal{L}_\xi) \Gamma_{MN}{}^P = Z^{PQ}{}_{RN} \partial_M \partial_Q \xi^R, \quad (5.1)$$

in order to manifest the inhomogeneous term. Tensors are characterised by  $\Delta_\xi = 0$ . This leads to the corresponding transformation of its derivative:

$$\begin{aligned} \Delta_\xi \partial_M \Gamma_{NP}{}^Q &= Z^{QR}{}_{SP} \partial_M \partial_N \partial_R \xi^S \\ &+ \Delta_\xi \Gamma_{MR}{}^Q \Gamma_{NP}{}^R - \Delta_\xi \Gamma_{MN}{}^R \Gamma_{RP}{}^Q - \Delta_\xi \Gamma_{MP}{}^R \Gamma_{NR}{}^Q. \end{aligned} \quad (5.2)$$

There are two possibilities to make the  $\partial^3 \xi$  terms vanish — antisymmetrisation  $[MN]$  or symmetrisation and projection on  $\overline{R}_2$ . We have not found any way of directly using the  $\overline{R}_2$  (although it will become clear below that it really is a specific combination of the two possibilities that leads to a tensor). Antisymmetrisation gives

$$\Delta_\xi (\partial_{[M} \Gamma_{N]P}{}^Q + \Gamma_{[M|P]}{}^R \Gamma_{N]R}{}^Q) = -\Delta_\xi \Gamma_{[MN]}{}^R \Gamma_{RP}{}^Q = \frac{1}{2} Y^{RS}{}_{TN} \Delta_\xi \Gamma_{SM}{}^T \Gamma_{RP}{}^Q, \quad (5.3)$$

where we have used the tensor property of the torsion of eq. (3.5) in the last step. This is a nice form that reduces to the covariant transformation of the Riemann tensor for ordinary geometry ( $Y = 0$ ). The middle step clearly shows why an attempt to construct a “Riemann tensor” fails, when the torsion-free condition does not suffice to set  $\Gamma_{[MN]}{}^P$  to zero. If however the expression on the right hand side of eq. (5.3) is contracted with  $\delta_Q^N$  and symmetrised in  $(MP)$ , it can be written as  $\Delta_\xi (\frac{1}{4} Y^{RS}{}_{TQ} \Gamma_{SM}{}^T \Gamma_{RP}{}^Q)$ . Therefore,

$$\begin{aligned} R_{MN} &= \partial_{(M} \Gamma_{|P|N)}{}^P - \partial_P \Gamma_{(MN)}{}^P \\ &+ \Gamma_{(MN)}{}^Q \Gamma_{PQ}{}^P - \Gamma_{P(M}{}^Q \Gamma_{N)Q}{}^P - \frac{1}{2} Y^{PQ}{}_{RS} \Gamma_{PM}{}^S \Gamma_{QN}{}^R \end{aligned} \quad (5.4)$$

transforms as a tensor. If we restrict to vanishing torsion, the last term may be rewritten using eq. (3.6), and the curvature takes the form

$$\begin{aligned} R_{MN} &= \partial_{(M} \Gamma_{|P|N)}{}^P - \partial_P \Gamma_{(MN)}{}^P \\ &+ \Gamma_{(MN)}{}^Q \Gamma_{PQ}{}^P - \frac{1}{2} \Gamma_{PM}{}^Q \Gamma_{QN}{}^P - \frac{1}{2} \Gamma_{P(M}{}^Q \Gamma_{N)Q}{}^P. \end{aligned} \quad (5.5)$$

An alternative way of deriving curvature is to start from the covariant constancy of the generalised vielbein, eq. (4.1). The procedure is to act with one more covariant derivative, and use only combinations where second derivatives on the vielbein are absent, due to either antisymmetry or the section condition. The result (which of course is zero) should be expressible as the difference of two tensors, of which the one expressed in terms of  $\Omega$  should be manifestly a tensor, and the one expressed in  $\Gamma$  manifestly invariant under local transformations in  $H$ . Then the equality of the two expressions implies that each of them enjoys the property manifest in the other.

Acting with a second derivative on eq. (4.1) gives

$$\begin{aligned} 0 = & \partial_M \partial_N E_P^A + \partial_M \Gamma_{NP}^Q E_Q^A - E_P^B \partial_M \Omega_{NB}^A \\ & - (\Gamma_N \Gamma_M)_P^Q E_Q^A - E_P^B (\Omega_M \Omega_N)_B^A + 2(\Gamma_{(M} E \Omega_{N)})_P^A . \end{aligned} \quad (5.6)$$

Antisymmetrising in  $[MN]$  gives

$$\begin{aligned} 0 = & (\partial_{[M} \Gamma_{N]} + \Gamma_{[M} \Gamma_{N]})_P^Q E_Q^A \\ & - E_P^B (\partial_{[M} \Omega_{N]} + \Omega_{[M} \Omega_{N]})_B^A , \end{aligned} \quad (5.7)$$

exactly as in ordinary geometry. The expression  $\partial_{[M} \Omega_{N]}$  on the second line is however not a tensor, since  $\Gamma_{[MN]}^P$  is not torsion. One has to form some combination of terms so that the  $\Gamma \Omega$  terms in eq. (5.6) combine with the  $\partial \Omega$  terms into covariant derivatives  $D^{(\Gamma)}$ . They can then be converted into  $\Omega$  using  $D_M A_N = E_N^A D_M A_A$ . This can be achieved with one contraction of indices and symmetrisation in the remaining two (as in the construction of the curvature above)<sup>3</sup>. The resulting curvature is identical to the one given in eq. (5.4), and its expression in terms of  $\Omega$  is

$$\begin{aligned} R_{MN} = & E_{(M}^A \partial_{N)} \Omega_{BA}^B - E_{(M}^A E_{N)}^B E_C^P \partial_P \Omega_{AB}^C - \frac{1}{2} Y^{PA}{}_{B(M} E_{N)}^C \partial_P \Omega_{AC}^B \\ & + \Omega_{(MN)}^A \Omega_{BA}^B - \Omega_{AM}^B \Omega_{BN}^A \\ & - \frac{1}{2} Y^{AB}{}_{C(M} (\Omega_{|AB}^D \Omega_{D|N)}^C + \Omega_{|A|N)}^D \Omega_{BD}^C) . \end{aligned} \quad (5.8)$$

(Here, we have used vanishing torsion and restricted the calculation to  $n \leq 6$ . We have also converted indices with the vielbein.)

We do not have a direct proof that  $R_{MN}$  exhausts the possible curvature tensors, although we suspect that this is the case. It is however clear that it is large enough to

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<sup>3</sup> Hohm and Zwiebach manage to form a 4-index tensor in the  $O(d, d)$  situation, where one has access to an  $H$ -invariant metric [11]. We do not see how that construction generalises to the exceptional cases.

contain anything we need. For example,  $R_1$  contains a 2-form in  $n$  dimensions, so there is enough room in  $R_{MN}$  for the modules of an ordinary Riemann tensor.

An important question is to what extent this curvature is defined in terms of a vielbein. This especially concerns its projection on  $\mathfrak{g}/\mathfrak{h}$ , since that part is a candidate for a “Ricci” or “Einstein” tensor, providing equations of motion for the geometry. A variation of the curvature gives at hand that

$$\delta R_{MN} = D_{(M} \delta \Gamma_{|P|N)}^P - D_P \delta \Gamma_{(MN)}^P . \quad (5.9)$$

There is nothing here that prevents the undefined module  $\Sigma$  from appearing in the second term. But if we consider the projection on  $\mathfrak{g}/\mathfrak{h}$ , we observe that  $(\mathfrak{g}/\mathfrak{h}) \otimes R_1 \not\supset \Sigma$ , so the variation of  $R_{MN}$  does not contain the  $\Sigma$  part of  $\delta \Gamma$ . Thus,  $R_{\{MN\}}$ , the projection of  $R_{MN}$  on  $\mathfrak{g}/\mathfrak{h}$ , is well-defined, and can serve as a Ricci tensor<sup>4</sup>.

From this it is also clear that the singlet, the curvature scalar  $R = G^{MN} R_{MN}$  (which is part of  $R_{\{MN\}}$ ), is well-defined in terms of the metric.

It is tempting to think of the curvature scalar as a Lagrangian for generalised gravity, whose variation should give an Einstein tensor. This of course has to rely on partial integration, since

$$\delta R = \delta(G^{MN} R_{MN}) = \delta G^{MN} R_{MN} + D_M (\delta \Gamma_N^{MN} - \delta \Gamma_N^{NM}) . \quad (5.10)$$

The  $D\delta\Gamma$  terms cannot be discarded unless the expression is multiplied by a scalar density from the measure, and it follows from eq. (4.6) that this density must have weight  $9 - n$ . So, if the Lagrangian density is

$$\mathcal{L} = |G|^{-\frac{9-n}{2|R_1|}} R , \quad (5.11)$$

the equations of motion for  $G_{MN}$ , the generalised Einstein’s equations, become

$$R_{\{MN\}} + \frac{9-n}{2|R_1|} G_{MN} R = 0 . \quad (5.12)$$

---

<sup>4</sup> The independence of the  $\Sigma$  part of  $\Gamma$  cannot be observed by simply entering an expression for  $\Gamma$  in terms of its decomposition in  $H$ -modules into eq. (5.4). Then the  $\Gamma\Gamma$  part of the second term would seem to contain  $\Sigma$ . One has to realise that the  $H$  subgroup defined by the vielbein/metric is special; only for this subgroup the covariant derivatives respect the decomposition into  $H$ -modules. We have checked in a couple of examples ( $n = 4, 5$ ) that an explicit decomposition in  $H$  modules yields no  $\Sigma^2$  in the  $\mathfrak{g}/\mathfrak{h}$  part, but indeed terms linear in  $\Sigma$ .

For pure generalised gravity, this is of course equivalent to  $R_{\{MN\}} = 0$ , but in presence of matter fields, as in the following section, eq. (5.12) provides the left hand side of the generalised Einstein’s equations.

We note that our density  $|G|^{-\frac{9-n}{2|R_1|}}$  agrees with the one given in ref. [35] for  $n = 4$ . There, the density is written as “ $M^{-1}$ ”, where  $M$  is the determinant of a metric on the fundamental **5** of  $SL(5)$ . We have  $-\frac{9-n}{2|R_1|} = -\frac{1}{4}$ , but our  $G_{MN}$  is a metric on the module **10**. The double weight of  $G$  and the double size of the determinant together account for the factor 4 compared to ref. [35].

## 6. TENSOR FIELDS

It is well known that the  $k$ -form gauge fields in dimensionally reduced theories come in the modules  $R_k$  under the U-duality group. Here, we instead ask for the dynamics in the “internal” directions, *i.e.*, for the descriptions of fields in  $R_k$  on a generalised manifold (at least locally). We need to be able to describe gauge symmetry and field equations, as well as some counting of degrees of freedom. The resulting description provides the U-duality version of the spinor of Ramond–Ramond fields for T-duality and double field theory [37].

The sequences  $\{R_k\}$  are symmetric under  $R_k \leftrightarrow \overline{R}_{9-n-k}$  (and the proper reassignment of  $\mathbb{R}$  weight), in analogy with forms. When we occasionally talk about modules  $R_k$  outside the window  $1 \leq k \leq 8 - n$ , which *e.g.* are needed for the complete reducibility, we will take the ones for  $k \geq 9 - n$  to agree with the ones given in ref. [34], which agrees with the positive levels of a Borcherds algebra [38] (the precise reason for this will be the subject of a future publication [39]). For  $k \leq 0$ , we will assume that the symmetry around  $k = \frac{9-n}{2}$  remains. Seen as objects with  $k$  upper indices, entities  $F^{M_1 \dots M_k}$  in  $R_k$  are in general neither totally antisymmetric nor symmetric, but have mixed symmetry.  $R_2$  is always symmetric, but already  $R_3$  is a module of mixed symmetry  $\boxplus$ .

In ref. [34] it was shown how the  $R_k$ ’s arise as an infinite sequence of ghosts related to the generalised diffeomorphisms and its reducibility. An essential property is that a derivative,  $\partial : R_k \rightarrow R_{k-1}$ , is nilpotent, so the sequence forms a complex. With this knowledge, it seems natural that the same modules should be responsible for gauge transformations of tensor fields (and their reducibilities).

We will now proceed to show that the sequence of modules  $\{R_k\}_{k=1}^{8-n}$  in many respects plays a rôle similar to that of forms on an ordinary manifold. An important piece of information is to what extent the affine connection takes part in the covariantised operation  $D : R_k \rightarrow R_{k-1}$ . Ideally, we would want connection to be absent, and “ $D = \partial$ ”, in analogy with the situation for the exterior derivative on forms.

It turns out that the derivative from  $R_k$  to  $R_{k-1}$  is connection-free for  $2 \leq k \leq 8 - n$ . For some simple cases, like  $R_2 \rightarrow R_1$  ( $n \leq 6$ ), it is straightforward to show:

$$\begin{aligned} D_N W^{MN} &= \partial_N W^{MN} - \Gamma_{NP}^M W^{PN} - \Gamma_{NP}^N W^{MP} \\ &= \partial_N W^{MN} - \frac{1}{2(n-1)} (Y^{NP}_{RS} \Gamma_{NP}^M + Y^{MP}_{RS} \Gamma_{NP}^N) W^{RS} = 0, \end{aligned} \quad (6.1)$$

with the use of eq. (3.9). For  $R_3 \rightarrow R_2$ , the proof is more involved, and relies on the hook ( $\boxplus$ ) property of  $R_3$ . For higher  $k$  it is more convenient to use  $R_{9-n-k} = \overline{R}_k$  and to treat them as tensor densities. For example, the covariant derivative from  $\overline{R}_1$  with weight  $w$  to  $\overline{R}_2$  is ( $n \leq 6$ )

$$Y_{MN}{}^{PQ} D_P W_Q = Y_{MN}{}^{PQ} (\partial_P W_Q - \frac{8-n-w}{9-n} \Gamma_{RP}^R W_Q), \quad (6.2)$$

where eq. (3.9) has been used again, showing that the derivative  $R_{8-n} \rightarrow R_{7-n}$  is connection-free ( $n \leq 6$ ).

However, it is obvious from direct inspection that  $R_1 \rightarrow R_0$  and  $R_{9-n} \rightarrow R_{8-n}$  contain connection. Neither is it possible to make the complex finite by using singlets at  $k = 0$  and  $k = 9 - n$ ; the corresponding derivatives also contain connection. These singlets actually both take the rôle one would have wanted from the other: the derivative  $\mathbf{1} \rightarrow \overline{R}_1$  is connection-free for weight 0, and the divergence  $R_1 \rightarrow \mathbf{1}$ , as we have seen, is connection-free when the singlet has weight  $9 - n$ . In some sense, it looks as though we had an  $(9 - n)$ -dimensional manifold, but with an exterior derivative “acting the wrong way”. To some extent, it becomes clearer from the diagrams in Appendix B what happens. They depict the action of an ordinary derivative on the modules  $R_k$  decomposed into  $GL(n)$  modules. There are always two sequences containing forms. All sequences are finite, but the ones starting at  $R_1$  (or lower) or ending at  $R_{8-n}$  (or higher) consist of the tensor product of a complex of forms with some non-trivial  $GL(n)$  module.

The problematic situation at the limits of the connection-free window does not prevent us from describing gauge connections and their field strengths within the window. It makes it more complicated to describe a gauge field in  $R_1$  (more about this below), and it seems to obstruct a complete covariant description of the full reducibility of the gauge transformations at any  $k$ .

Consider a gauge field  $A$  in  $R_{k+1}$ ,  $1 \leq k \leq 7 - n$ . It will have a field strength  $F = \partial A$  in  $R_k$ . There is a gauge symmetry  $\delta_\Lambda A = \partial \Lambda$  with parameter  $\Lambda$  in  $R_{k+2}$  and a Bianchi identity  $\partial F = 0$  in  $R_{k-1}$ . (For  $k = 7 - n$  the above discussion shows a difficulty with the covariance of the gauge transformation, and similarly with the Bianchi identity for  $k = 1$ . We will for the moment ignore this issue.)

Given a metric, there is a natural duality operation, taking  $F$  in  $R_k$  to  $*F$  in  $R_{9-n-k}$ . This can be written in two ways (analogous to lower or upper indices for ordinary forms).

One is obtained by simply lowering the  $k$  indices with the metric. This results in a tensor in  $\overline{R}_k$  with weight  $-k$ . A tensor in  $R_{9-n-k}$  has weight  $9-n-k$ , so the weight has to be adjusted by an appropriate power of  $|G|$ . The correct dual field strength is

$$*F_{M_1 \dots M_k} = |G|^{-\frac{9-n}{2|R_1|}} G_{M_1 N_1} \dots G_{M_k N_k} F^{N_1 \dots N_k} . \quad (6.3)$$

The other way is to use an invariant tensor  $\Sigma^{A_1 \dots A_{9-n}}$ , which after conversion of indices with inverse vielbeins becomes a tensor  $\Sigma^{M_1 \dots M_{9-n}}$  and write

$$*F^{M_{k+1} \dots M_{9-n}} = \Sigma^{M_1 \dots M_{9-n}} G_{M_1 N_1} \dots G_{M_k N_k} F^{N_1 \dots N_k} . \quad (6.4)$$

The equation of motion for  $A$  can now be written

$$\partial *F = 0 . \quad (6.5)$$

Since only connection-free derivatives have been used for forming the field strengths and the equations of motion, it is clear that there are no problems with undefined connection. The metric enters only through the dualisation. There is a duality symmetry under  $k \rightarrow 9-n-k$  exchanging equations of motion and Bianchi identities. Again, we find that a Lagrangian density  $F * F$  with weight  $9-n$  is necessary in order to make partial integration possible.

It may seem that it is problematic to use a gauge potential in  $R_1$ , since the field strength would belong to  $R_0$ , which is outside the connection-free window. For a number of reasons (one is the field content of maximally supersymmetric generalised supergravity, see below) one would still like to have potentials in  $R_1$ . Although we will leave the detailed formulation to future work, we would like to argue that it is meaningful to have such a potential. The argument is based on dimensional reduction of generalised gravity. We will consider linearised fields. The linearised degrees of freedom of generalised gravity lie in  $\mathfrak{g}/\mathfrak{h}$ . Consider the decomposition under “dimensional reduction”, *i.e.*, when  $n$  is lowered by 1. We drop the singlet part, which is irrelevant for the argument, and do not consider the weights of resulting modules. Let us denote the module  $\mathfrak{e}_{n(n)}/\mathfrak{k}(\mathfrak{e}_{n(n)})$  by  $\phi_n$ . Under dimensional reduction,  $\phi_n \rightarrow \phi_{n-1} \oplus R_1^{(n-1)} \oplus \mathbf{1}$ . The  $R_1$  in the lower-dimensional theory is a “generalised graviphoton”, whose dynamics is dictated by generalised gravity in the higher dimension. We have not examined the details of this, but it clearly shows that one can have fields in  $R_1$ .

The following is also worth noticing about derivatives on  $R_1$ . Taking a derivative of a field  $A$  in  $R_1$  gives  $D_Q A^R = \partial_Q A^R - \Gamma_{QM}^R A^M$ . We can use the  $Z$ -tensor to pick out the  $\mathfrak{g}$  part:

$$Z^{PQ}{}_{RN} D_Q A^R = Z^{PQ}{}_{RN} (\partial_Q A^R - \Gamma_{QM}^R A^M) = Z^{PQ}{}_{RN} \partial_Q A^R + \Gamma_{MN}^P A^M, \quad (6.6)$$

where the torsion-free property was used for the second term. If the free index pair  ${}_N^P$  is projected on  $\mathfrak{g}/\mathfrak{h}$ , only well-defined connection enters. In addition, the  $\mathfrak{g}/\mathfrak{h}$  part of the compatibility equation (4.4) tells us that the  $\mathfrak{g}/\mathfrak{h}$ -valued part of a compatible  $\Gamma_M$  contains a  $\partial_M$  and obeys the section condition. Therefore, even if the derivative  $R_1 \rightarrow \mathfrak{g}/\mathfrak{h}$  contains connection, a field strength  $F = (DA)|_{\mathfrak{g}/\mathfrak{h}}$  allows for a gauge invariance with parameter in  $R_2$ . Such an invariance is expected, since  $R_1^{(n)} \rightarrow R_1^{(n-1)} \oplus R_2^{(n-1)} \oplus \mathbf{1}$  under dimensional reduction.

We would like to say some words about the counting of degrees of freedom, both off-shell and on-shell. The models we are dealing with are effectively euclidean field theories, so in a strict sense it is not meaningful to talk about local on-shell degrees of freedom. What we mean is the number of physical polarisations the on-shell fields would carry, had the model been formulated with another real form of  $G$  corresponding to Minkowski signature after solution of the section condition. This gives numbers that are of practical use, especially when it comes to supersymmetric models [33,40] and matching of bosonic and fermionic degrees of freedom.

The counting of off-shell degrees of freedom is straightforward. It is simply given by the number of field components subtracted with the number of gauge parameters. Here, the infinite reducibility has to be taken into account, and we thus know that the number of off-shell degrees of freedom of a gauge field in  $R_k$  is

$$N_k = \sum_{\ell=0}^{\infty} (-1)^\ell |R_{k+\ell}|. \quad (6.7)$$

Such sums are naïvely divergent (the terms are alternating but growing) but have a meaningful regularisation [41,34]. Of course, it is enough to perform the regularisation for  $N_1$  and calculated the finite difference. The result for  $1 \leq k \leq 8 - n$  is

$$N_k^{(n)} = \begin{cases} |R_k^{(n-1)}| + 1 & k = 1, \\ |R_k^{(n-1)}| & 2 \leq k \leq 8 - n. \end{cases} \quad (6.8)$$

The numbers have the property  $N_k^{(n)} = N_{10-n-k}^{(n)}$ .



The on-shell number of degrees of freedom can safely be deduced from the observation that all the fields on the  $n$ -dimensional solution of the section condition are forms (ordinary massless tensor fields). Therefore, the number of on-shell degrees of freedom of a field in  $R_k^{(n)}$  is obtained as  $|R_k^{(n-2)}|$ . The number of physical polarisations of a field is obtained by regarding the “same” field in “two dimensions less”, just as the counting goes for massless fields in Minkowski space. Since  $R_{k+1}^{(n-2)} = \overline{R}_{10-n-k}^{(n-2)}$ , this counting agrees with the dualisation from a potential for  $F$  in  $R_{k+1}^{(n)}$  to a potential for  $*F$  in  $R_{10-n-k}^{(n)}$ .

The counting has been tested on a number of non-gravitational supermultiplets [40]. Here we will illustrate it by counting the bosonic degrees of freedom in the maximal generalised supergravity. Fields will transform under the  $SL(11-n)$  or  $SO(1, 10-n)$  “R-symmetry” of the “reduced” directions, and behave as forms under these. If one associates  $R_k$  with a  $k$ -form for  $k = 1, \dots, [\frac{11-n}{2}]$ , and asks for a selfduality for  $R_{\frac{11-n}{2}}$  when  $n$  is odd, the resulting counting is as follows:

$n$	gen. gravity	scalar coset	$R_k$	total
4	2	28	$\binom{7}{1} \times 3 + \binom{7}{2} \times 2 + \binom{7}{3} \times 1 = 98$	128
5	6	21	$\binom{6}{1} \times 6 + \binom{6}{2} \times 3 + \frac{1}{2} \times \binom{6}{3} \times 2 = 101$	128
6	13	15	$\binom{5}{1} \times 10 + \binom{5}{2} \times 5 = 100$	128
7	24	10	$\binom{4}{1} \times 16 + \frac{1}{2} \times \binom{4}{2} \times 10 = 94$	128

Table 4: Counting of bosonic degrees of freedom for maximal supersymmetry.

Note that for  $n = 7$  also  $R_2 = R_{9-n} = \mathbf{133}$ , which we have not discussed above, is needed. Maybe the dual of the well-defined derivative  $R_1 \rightarrow \mathfrak{g}/\mathfrak{h}$  can be of use. The appearance of fields as forms in  $R_k$  is well known. In the present context it can also be obtained from dimensional reduction. We have already seen that the generalised gravity on dimensional reduction gives rise to a generalised graviphoton in  $R_1$ . The generic rule for tensor fields is that  $R_k^{(n)}$  gives rise to  $R_k^{(n-1)}$  and  $R_{k+1}^{(n-1)}$  (with an extra singlet for  $k = 1$  and  $k = 8 - n$ ), in close analogy to form fields. This is how the binomial coefficients are sequentially built.

## 7. CONCLUSIONS

We have presented a tensor calculus for exceptional generalised geometry. This includes universal and covariant expressions for connections and curvatures. Our analysis agrees with ref. [32], but has manifest covariance, and with ref. [35] for  $n = 4$ . We have also given details on tensor gauge fields and their coupling to exceptional geometry. Some technical

issues remain concerning the “generalised graviphoton” field. Even if the local description in terms of a tensor calculus respecting infinitesimal transformation now seems complete, important questions concerning the concept of generalised manifolds remain open. Hohm and Zwiebach have discussed the issue of exponentiating the Lie derivative in double field theory to a large diffeomorphism, but there are many remaining questions. An important one is to find an integration measure.

In ref. [33], minimal exceptional supergravity was formulated. In an accompanying paper [40] non-gravitational supermultiplets based on the tensor fields we present here were constructed. Extended supergravity will demand inclusion of such multiplets. It would be very interesting to investigate the possibility of formulating such models as some generalised supergeometry. It is not clear which set of modules will accompany the  $R_k$ ’s in order to build the correspondence to “forms on superspace”. Such a formulation, preferably in an off-shell version using pure spinor techniques generalising refs. [42,43], could perhaps provide a simultaneous manifestation of supersymmetry and U-duality.

Note added: The paper [44], which appeared on the completion of our work, specialises on  $n = 7$  and has a substantial overlap with the present paper concerning the geometric analysis.

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## APPENDIX A: NOTATION

$G$  and  $H$  denote throughout the paper the groups  $G = E_{n(n)} \times \mathbb{R}^+$  and its compact subgroup  $H = K(E_{n(n)})$ , and their Lie algebras (and adjoint modules) are written  $\mathfrak{g}$  and  $\mathfrak{h}$ . For the complement to  $\mathfrak{h}$  in  $\mathfrak{g}$  we use “ $\mathfrak{g}/\mathfrak{h}$ ” (even if “ $\mathfrak{g} \ominus \mathfrak{h}$ ” might have been more correct). A projection of a 2-index object on  $\mathfrak{g}/\mathfrak{h}$  is denoted by curly brackets:  $\{MN\}$ .

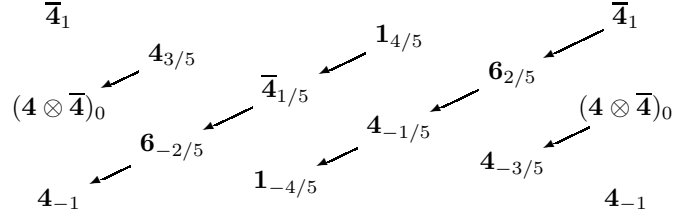
We use the notation  $\vee$  for symmetrised tensor product. The dimension of a module  $R$  is denoted  $|R|$ . When a module in the sequence  $\{R_k\}$  carries an upper index,  $R_k^{(n)}$ , it refers to  $n$ , the rank of the exceptional group.

APPENDIX B: THE ACTION OF A DERIVATIVE AMONG THE  $R_k$ 

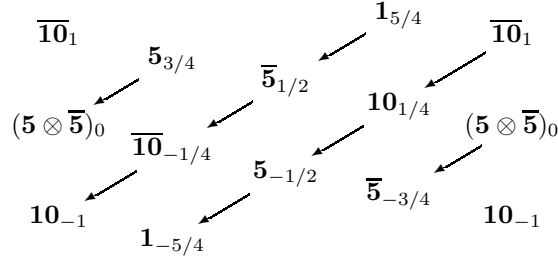
Below are diagrams showing the action of a derivative fulfilling the section condition on elements in  $R_k$ ,  $0 \leq k \leq 9 - n$ . The modules are split into modules of  $SL(n) \times \mathbb{R}$ . For  $n = 6, 7$ , there is an  $SL(2)$  which is broken to  $\mathbb{R}$  by the solution of the section condition.

Note that there are always two lines containing the ordinary  $n$ -dimensional forms. Other lines consist of the tensor product of the forms by some non-trivial module. Such lines begin at  $R_1$  and end at  $R_{8-n}$ , and may be seen as responsible for the appearance of connection.

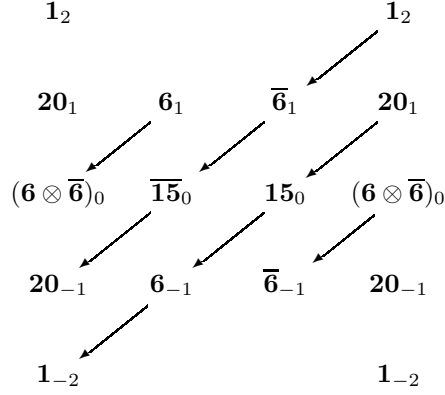
$n = 4$ :



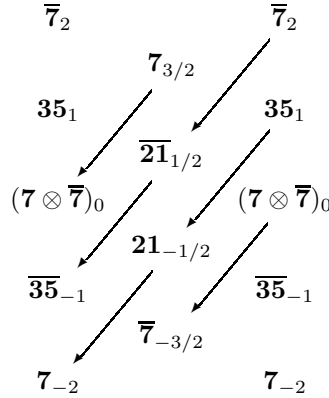
$n = 5$ :



$n = 6$ :



$n = 7$ :



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